M.Sc (Sem 03). Paper 11
Unit - 03.

Complex linear spaces and normed linear spaces (Examples)

Example 1: Let N be a real normed linear space and suppose $f(x)=0$ for all $f \in N^*$. Show that $x=0$.

Proof: Suppose $x \neq 0$. Then, there exists $f \in N^*$ such that $f(x) = \|x\| > 0$ which contradicts the hypothesis that $f(x)=0$ for all $f \in N^*$. Hence we must have $x=0$.

Example 2. Let M be a closed linear subspace of a normed linear space N and let x_0 be a vector not in M. If d is the distance from x_0 to M. Show that there exists a functional $F \in N^*$ such that $F(M)=\{0\}$, $F(x_0)=d$ and $\|F\|=1$.

Solution: First we have by definition.

$$d = \inf\{\|x_0 - x\| : x \in M\}.$$

Since M is closed and $x \notin M$: $d > 0$.

Now consider the subspace $M_0 = \{x + \alpha x_0 : x \in M \ \alpha \text{ real}\}$ spanned by M and x_0. Since $x_0 \notin M$, the representation of each vector y in M_0 in the form $y = x + \alpha x_0$ is unique.

Define the map f_0 on M_0 by

$$f_0(y) = \alpha d$$

Where $y = x + \alpha x_0$ and d as in the hypothesis. Because of the uniqueness of y, the mapping f_0 is well defined. It is clear that f_0 is linear on M_0. Also $f_0(x_0) = f_0(0 + 1 \cdot x_0)$ $= 1 \cdot d = d$ and if $m \in M$, then $f_0(m) = f_0(m + 0 \cdot x_0) = 0 \cdot d = 0$ so that $f_0(M) = \{0\}$. We now prove that $\|f_0\| = 1$.

We have $\|f_0\| = \sup\left\{\dfrac{|f_0(y)|}{\|y\|} : y \in M_0, y \neq 0\right\}$.

$$= \sup\left\{\frac{|f_0(x+\alpha x_0)|}{\|x+\alpha x_0\|} : x \in M, \alpha \in R, x \neq 0 \text{ or } \alpha \neq 0\right\}$$

$$= \sup\left\{\frac{|\alpha d|}{\|x+\alpha x_0\|} : x \in M, \alpha \in R, \alpha \neq 0\right\}.$$

$[\because F(y) = 0 \text{ when } \alpha = 0]$

$$= \sup\left[\frac{d}{\|x_0 + \frac{x}{\alpha}\|} : x \in M, \alpha \in R, \alpha \neq 0\right]$$

$[\because d > 0, |\alpha d| = d|\alpha|]$

$$= d \sup\left\{\frac{1}{\|x_0 - z\|} : z = -x/\alpha \in M\right\}.$$

$$= d\left[\inf\{\|x_0 - z\| : z \in M\}\right]^{-1} = d \cdot \frac{1}{d} = 1 \text{ by (1)}$$

Thus f_0 is a linear functional on M_0 such that

$$f_0(M) = \{0\},\ f_0(x_0) = d \text{ and } \|f_0\| = 1 \quad \text{——(2)}$$

Hence by the Hahn–Banach theorem there exists a functional F on the whole space N such that

$$F(y) = f_0(y)\ \forall\, y \in M_0 \text{ and } \|F\| = \|F_0\|.$$

It follows from (2) that

$$F(M) = \{0\},\ F\{x_0\} = d \text{ and } \|f\| = 1.$$

Proved;

Example-03. Prove that a normed linear space is separable if its conjugate (or dual) space is separable.

Solution:— Let N be a normed linear space whose conjugate space N^* is separable. Consider the set

$$S = \{f : f \in N^*, \|f\| = 1\}.$$

Since every subspace of a metric space is separable, ~~S must~~ must be separable. Hence by definition of separability, S contains countable dense subset, say $A = \{f_1, f_2, \text{——}, f_n \text{——}\}$.

Since each $f_n \in S$, we have $\|f_n\| = 1$ for all n.

Since $\|f_n\| = \sup\{|f_n(x)| : \|x\| = 1\}$, for each n, there must exist some vector x_n with $\|x_n\| = 1$ such that

$$|f_n(x_n)| > \frac{1}{2}.$$

(If such x_n did not exist, this would contradict the fact that $\|f_n\| = 1$).

Let M be the closed linear subspace in N generated by the sequence $\langle x_n \rangle$. We assert that $M = N$. Suppose $M \neq N$ and let $x_0 \in N - M$ ie $x \notin M$ by above example, there exists an $F \in N^*$ such that

$$\|F\| = 1,\ F(x_0) \neq 0 \text{ and } F(x) = 0 \text{ if } x \in M$$

Since $\|F\| = 1$, $F \in S$ and since each $x_n \in M$, we have

$$F(x_n) = 0 \quad (n = 1, 2, 3, \text{——}). \text{ Now}$$

$$\begin{aligned}
\frac{1}{2} < |f_n(x_n)| &= |(f_n(x_n) - F(x_n)) + F(x_n)| \\
&\leq |f_n(x_n) - F(x_n)| + |F(x_n)| \\
&= |(f_n - F)(x_n)| \qquad (\because F(x_n) = 0) \\
&\leq \|f_n - F\| \|x_n\| = \|f_n - F\| \quad (\because \|x_n\| = 1)
\end{aligned}$$

Thus $\|f_n - F\| \geq \frac{1}{2}$ for all n.

Now, since A is dense in S, every point of S is an adherent point of A so that each sphere centred at arbitrary $F \in S$ must contain a point of A. But the open sphere $\{f : \|f - F\| < \frac{1}{2}\}$ centred at $F \in S$ contains no point of A. We thus arrive at a contradiction and so we must have $M = N$. It then follows that the set of all linear combinations of the x_n's whose coefficients are rational or if N is complex have rational real and imaginary parts, constitute a countable set everywhere dense in N and consequently N is separable.

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